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LAGRANGIAN COORDINATES FOR A DRIFT–FLUX MODEL OF A GAS–LIQUID MIXTURE

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Abstract—The mass Lagrangian coordinate associated with the velocity field of the gas for a nonstationary drift–flux model of a gas–liquid mixture is introduced here. If the mass transfer is neglected, this results in a quite simple structure of the governing equations and permits to integrate one of the equations independently on the others. The simplified model, describing the horizontal two-phase flow for high Froude number is considered and some explicit solutions to this model in the mass Lagrangian coordinates are obtained. Copyright © 1996 Elsevier Science Ltd.

Key Words: drift-flux model, mass Lagrangian coordinates, explicit solutions

1. INTRODUCTION

Two-phase flow modelling is typically based on averaging the instanteneous local equations of motion (Ishii 1975, Delhaye & Achard 1976, Nigmatulin 1979, Drew 1983 and others). After formal averaging, there remains the non-trivial task of formulating closure relations in terms of averaged variables. Generally, the closure relations are specific to the structure of the flow (bubbly flow, slug flow, separated flow etc.) and have to be separately supplied for each flow pattern. For investigation of unsteady phenomena in gas–liquid pipe flow Fabre *et al.* (1989) proposed using a simple one-dimensional nonstationary model consisting of two equations for mass conservation and the equation of conservation of the total momentum. These equations were supplied with an empirical relation between the mean gas velocity and the mean volumetric flux—"mean" denotes hereinafter cross-section averaging. This relationship is similar to that of Zuber & Findley (1965), Bendiksen (1984), França & Lahey (1992) and others. This transient drift–flux model was tested numerically and the results gave a satisfactory agreement with the experiments (Caussade *et al.* 1989, Fabre *et al.* 1995). One of the most important properties of this model is that it is hyperbolic in a physically reasonable region of parameters (Théron 1989, Benzoni-Gavage 1991). In this paper we propose to use mass Lagrangian coordinates to simplify the equations of drift–flux model. These coordinates have been used very often with the one-dimensional gas dynamics equations to solve free boundary value problems. Using mass Lagrangian coordinates the entropy equation can be integrated independently. However, in the case of shocks, this integration is invalid because the energy equation is not equivalent to the entropy equation through the shock: the entropy increases across the shock, whereas the energy is conserved. We will prove here that this procedure is correct for the drift–flux model, if we introduce the mass Lagrangian coordinates associated with the gas velocity. The drift–flux model is described in section 2. In section 3 the mass Lagrangian coordinates are introduced and the model is rewritten in an equivalent form. A simplified model describing non-stationary gas–liquid flow in horizontal pipe for high Froude number is formulated in section 4. Explicit solutions of the simplified model are presented in section 5.

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2. GOVERNING EQUATIONS

A simple model describing the flow of the gas–liquid mixture in an inclined pipe consists of two equations of mass conservation and the equation of the total momentum:

$$(\rho_L \epsilon_L)_t + (\rho_L \epsilon_L u_L)_x = \dot{m}_L, \qquad [1]$$

$$(\rho_G \epsilon_G)_t + (\rho_G \epsilon_G u_G)_x = \dot{m}_G, \qquad [2]$$

$$(\rho_L \epsilon_L u_L + \rho_G \epsilon_G u_G)_t + (\rho_L \epsilon_L u_L^2 + \rho_G \epsilon_G u_G^2 + p)_x = \rho_M\left(-g \sin\theta - \frac{2f}{D} u_M |u_M|\right). \qquad [3]$$

Here t is the time; x is the space coordinate along the pipe; ρ, ϵ, u and $\dot{m}$ are the phase density, the mean void fraction, the mean velocity and the mass by unit of volume entering into the phase across the interface, respectively. Index L is refered to the liquid and index G to the gas. The void fractions satisfy the geometric relation $\epsilon_G + \epsilon_L = 1$, and the mass fluxes the mass conservation $\dot{m}_G + \dot{m}_L = 0$. D is the diameter of the pipe; f is the friction coefficient; g is the gravity; θ is the angle of the inclination of the pipe. We define also the mean density and mean velocity of the mixture:

$$\rho_M = \rho_G \epsilon_G + \rho_L \epsilon_L, \quad u_M = u_G \epsilon_G + u_L \epsilon_L. \qquad [4]$$

The system is closed if we add constitutive laws

$$u_G = c_0 u_M + u_\infty. \qquad [5]$$

$$p = \rho_G r_G T, \quad \rho_L = \text{const}. \qquad [6]$$

Equation [5] has a broad validity since it has been used with a similar form for both bubbly flow and slug flow, and it is well established for a wide range of parameters (Zuber & Findlay 1965, Bendiksen 1984). The following two typical sets of values of c_0 and u_∞ are well accepted for slug flow (Bendiksen 1984):

$$c_0 = 1.05 + 0.15 \sin^2\theta, \qquad [7]$$

$$u_\infty = (gD)^{1/2}(0.35 \sin\theta + 0.54 \cos\theta) \quad \text{for} \quad \frac{|u_M|}{(gD)^{1/2}} < 3.5$$

or

$$c_0 = 1.2, \quad u_\infty = (gD)^{1/2} 0.35 \sin\theta \quad \text{for} \quad \frac{|u_M|}{(gD)^{1/2}} > 3.5.$$

In [6] r_G is the constant of the perfect gas, T is the temperature of the gas. We shall assume also that the temperature is constant. Isothermal assumption implies that both phases are in the thermal equilibrium, the liquid imposing its temperature to the gas. This simplification avoids to treat the equation of energy. Both the mass flux $\dot{m}_G$ and the friction factor f are unknown quantities. They have to be closed by physical relations. It is generally acknowledged that these quantities can depend on velocity, void fraction and physical properties, but not on their derivatives. Théron (1989) has proven hyperbolicity of the system [1]–[6] in the limit where the terms $\rho_G \epsilon_G u_G$ and $\rho_G \epsilon_G u_G^2$ in [3] are much smaller than $\rho_L \epsilon_L u_L$ and $\rho_L \epsilon_L u_L^2$, respectively. He found that there exists a critical void fraction $\epsilon_G^* = 1/c_0$ below which the simplified system remains hyperbolic with the following slopes of characteristics $\lambda_i = dx/dt$, $i = 1, 2, 3$:

$$\lambda_{1,2} = u_L \pm \left(\frac{p}{\rho_L \epsilon_G (1 - c_0 \epsilon_G)}\right)^{1/2}, \quad \lambda_3 = u_G.$$

Mathematical properties of the system [1]–[3] were studied by Benzoni-Gavage (1991). She has formulated the hyperbolicity conditions and has shown (in the case of Théron) that the first two characteristic fields of the system are genuinely nonlinear in the sense of Lax, while the last one is a linearly degenerate field (see Lax 1957 for definitions and section 5 of this paper). In the very much simplified situation where $\dot{m}_L = \dot{m}_G = 0, f = 0, u_\infty = 0$ and $c_0 = 1$, the system [1]–[3] is reduced to:

$$(\epsilon_L)_t + (\epsilon_L u_L)_x = 0, \qquad [1']$$

$$(c\epsilon_L)_t + (c\epsilon_L u_L)_x = 0, \qquad [2']$$

$$((1+c)\epsilon_L u_L)_t + ((1+c)\epsilon_L u_L^2 + \pi)_x = 0, \qquad [3']$$

where

$$c = \frac{\rho_G \epsilon_G}{\rho_L \epsilon_L}, \quad \pi = r_G T \frac{c\epsilon_L}{1-\epsilon_L}.$$

The system [1′]–[3′] resembles the gas dynamics equations. Hence, introducing here the mass Lagrangian coordinate ζ through the change of variables

$$\frac{\partial x}{\partial \zeta} = V \equiv \frac{1}{\epsilon_L}, \quad \frac{\partial x}{\partial t} = U \equiv u_L,$$

the system is reduced to the following:

$$V_t - U_\zeta = 0, \qquad [1'']$$

$$c_t = 0, \qquad [2'']$$

$$((1+c)U)_t + \pi_\zeta = 0. \qquad [3'']$$

The system [1″]–[3″] is equivalent to the system [1]–[3] not only for continuous solutions, but also for the shocks. Furthermore, it has some advantages. First, it is simpler. Second, the equation [2″] can be integrated independently on [1″], [3″] and this greatly simplifies the numerical procedure. Unfortunately, the system [1′]–[3′] is never used in practice because, typically, u_∞ and $c_0 - 1$ are not equal to zero simultaneously. Further, we introduce Lagrangian coordinates for the system [1]–[3] in the general situation and consider corresponding physically relevant simplified models and their explicit solutions.

3. LAGRANGIAN COORDINATES

Recalling [5] and the definition of the mean velocity u_M [4], we obtain

$$u_G = \frac{\epsilon_L u_L}{\epsilon_L - \epsilon_L^*} + \frac{(1-\epsilon_L^*)u_\infty}{\epsilon_L - \epsilon_L^*}, \qquad [8]$$

$$u_M + (1-\epsilon_L^*)(u_G - u_\infty) = \epsilon_G^*(u_G - u_\infty), \qquad [9]$$

with the definitions of critical phase fractions

$$\epsilon_L^* = \frac{1}{c_0}, \quad \epsilon_L^* = \frac{c_0 - 1}{c_0} \geqslant 0, \quad \epsilon_L^* + \epsilon_G^* = 1. \qquad [10]$$

It follows from [8] that

$$\epsilon_L u_L = (\epsilon_L - \epsilon_L^*)u_G - (1-\epsilon_L^*)u_\infty. \qquad [11]$$

Substituting [11] into [1] and taking into account that ρ_L and u_∞ are constant, we get

$$(\rho_L(\epsilon_L - \epsilon_L^*))_t + (\rho_L(\epsilon_L - \epsilon^*)u_G)_x = \dot{m}_L. \qquad [12]$$

We introduce now the "pseudo density" ρ (see the definition [4] of the average density ρ_M):

$$\rho = \rho_L(\epsilon_L - \epsilon_L^*) + \rho_G \epsilon_G \equiv \rho_M - \epsilon_L^* \rho_L \qquad [13]$$

and the "pseudo mass concentration" c_L of the liquid

$$c_L = \frac{\rho_L(\epsilon_L - \epsilon_L^*)}{\rho}. \qquad [14]$$

The "pseudo density" ρ and the "pseudo mass concentration" c_L are different from the mixture density ρ_M and the liquid mass concentration $\epsilon_L\rho_L/\rho_M$. However, in some sense they are similar: as ρ_M varies between ρ_G and ρ_L, and $\epsilon_L\rho_L/\rho_M$ between 0 and 1, $\rho/(1-\epsilon_L^*)$ and c_L do the same if and only if ϵ_L varies between ϵ_L^* and 1. In the following, we consider only this case. In fact, the opposite case $\epsilon_L < \epsilon_L^*$ corresponds more or less to the annular flow pattern when the drift–flux model is not applicable. Moreover, c_L is no longer positive and ρ can be negative. The system [1]–[3] with the conditions [5]–[6] taken into account is equivalent to

$$(\rho c_L)_t + (\rho c_L u_G)_x = \dot{m}_L, \qquad [15]$$

$$(\rho)_t + (\rho u_G)_x = 0, \qquad [16]$$

$$(\rho u_G)_t + (\rho u_G^2 + p + \rho_L\epsilon_L u_L^2 - \rho_L(\epsilon_L - \epsilon_L^*)u_G^2)_x = \rho_M\left(-g\sin\theta - \frac{2f}{D}u_m|u_M|\right). \qquad [17]$$

When $\dot{m}_L = 0$ it follows from [15], [16] that

$$(c_L)_t + u_G(c_L)_x = 0,$$

i.e. c_L is a Riemann invariant (see Lax 1957 for definitions). This fact was discovered earlier by Fabre *et al.* (1990) and further by Benzoni-Gavage (1991) in different variables. It is quite easy to express the pressure p in terms of c_L and ρ. Indeed, it follows from [6], [13] that

$$\rho = \rho_L(\epsilon_L - \epsilon_L^*) + \frac{p\epsilon_G}{r_G T}.$$

Dividing this identity by ρ and recalling the definition [14] of c_L we obtain

$$p = \frac{r_G T}{1-\epsilon_L^*}\frac{(1-c_L)\rho}{1 - \dfrac{c_L\rho}{\rho_L(1-\epsilon_L^*)}}. \qquad [18]$$

Now, we can introduce the Lagrangian coordinate ξ instead of x, where $x = x(t, \xi)$ is defined from the solution of the Cauchy problem:

$$\frac{dx}{dt} = u_G(t, x), \quad x(0, \xi) = \xi.$$

For any function $F(t, x)$ we define $\tilde{F}(t, \xi) = F(t, x(t, \xi))$. Then, for $F = \rho$ the equation [16] is reduced to

$$\tilde{\rho}(t, \xi)\frac{\partial x}{\partial \xi} = \tilde{\rho}(0, \xi). \qquad [19]$$

If we introduce the mass Lagrangian coordinate

$$q = \int_0^\xi \tilde{\rho}(0, \xi)\, d\xi,$$

we transform [19] to the following form:

$$\frac{\partial x}{\partial q} = \tilde{v}, \quad \tilde{v} = \frac{1}{\tilde{\rho}}. \qquad [19']$$

Hence, taking into account [19′], we obtain from [15]–[17]

$$(\tilde{c}_L)_t = \dot{m}_L\tilde{v}, \qquad [20]$$

$$\tilde{v}_t - (\tilde{u}_G)_q = 0, \qquad [21]$$

$$(\tilde{u}_G)_t + (\tilde{p} + \rho_L\tilde{\epsilon}_L(\tilde{u}_L^2 - \tilde{u}_G^2) + \rho_L\epsilon_L^*\tilde{u}_G^2)_q = \tilde{\rho}_M\tilde{v}\left(-g\sin\theta - \frac{2f}{D}\tilde{u}_M|\tilde{u}_M|\right). \qquad [22]$$

The system [18], [20]–[22] is equivalent to the system [15]–[17] even for shocks. Typically, $\tilde{\dot{m}}_L = \mu \tilde{M}_L$, where μ is a small parameter, and $\tilde{M}_L$ is a bounded function. This means that the system [20]–[22] is decoupled for the numerical procedure: thus for explicit schemes the Courant–Friedrichs–Lewy condition needs to be used only for the subsystem [21]–[22]. From now, we will drop the tilde over the dependent variables.

4. SIMPLIFIED EQUATIONS

We consider here a practically important case (see [7]) when

$$\epsilon_L^* \neq 0, \; u_\infty = 0,$$

which corresponds to a motion of a two-phase mixture in a horizontal pipe ($\theta = 0$) at high Froude number; it is typically used for slug flow (see survey by Fabre & Liné 1992). It follows from [8] that

$$\epsilon_L(u_L - u_G) = -u_G \epsilon_L^* - u_\infty(1 - \epsilon_L^*),$$

$$\epsilon_L(u_L - u_G) = u_G(2\epsilon_L - \epsilon_L^*) - u_\infty(1 - \epsilon_L^*).$$

Combining the formulae above, leads to

$$\epsilon_L(u_L^2 - u_G^2) = -\frac{1}{\epsilon_L}(u_G\epsilon_L^* + u_\infty(1 - \epsilon_L^*))(u_G(2\epsilon_L - \epsilon_L^*) - u_\infty(1 - \epsilon_L^*)). \qquad [23]$$

If $u_\infty = 0$ then [23] implies that

$$\epsilon_L(u_L^2 - u_G^2) = -2u_G^2\epsilon_L^* + u_G^2(\epsilon_L^*)^2/\epsilon_L. \qquad [24]$$

Taking into account [24], we get from [18], [20]–[22] that

$$(c_L)_t = \dot{m}_L v, \qquad [25]$$

$$v_t - (u_G)_q = 0, \qquad [26]$$

$$(u_G)_t + (P(c_L, v, u_G))_q = -\rho_M v \frac{2f}{D} u_M |u_M|, \qquad [27]$$

where the modified pressure P is defined by

$$P(c_L, v, u_G) = p(c_L, v) - \frac{c_L u_G^2}{v + \dfrac{c_L}{\rho_L \epsilon_L^*}}, \quad p(c_L, v) = \frac{r_G T}{1 - \epsilon_L^*} \frac{1 - c_L}{v - \dfrac{c_L}{\rho_L(1 - \epsilon_L^*)}} \qquad [28]$$

and u_M is determined by [9] with $u_\infty = 0$. If we neglect both friction force ($f = 0$) and mass transfer ($\dot{m}_L = 0$) we obtain equations similar to the one-dimensional gas dynamics equations. But in the present equations c_L does not play the same role as the entropy in the gas dynamics equations because it does not increase through the shock. Moreover, the "pressure" P depends here not only on the "pseudo mass concentration" c_L and the specific volume v, but also on the velocity u_G. Transforming the system [25]–[27] to the matrix form

$$\mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_q = \mathbf{f},$$

where $\mathbf{u} = (c_L, v, u_G)^{Tr}$ and $\mathbf{f}$ is the r.h.s. of [25]–[27], we can calculate the eigenvalues of $\mathbf{A}(\mathbf{u})$:

$$\lambda_1 = 0,$$

$$\lambda_{2,3} = -\frac{c_L u_G}{v + \dfrac{c_L}{\rho_L \epsilon_L^*}} \pm \left(-\frac{\partial p(c_L, v)}{\partial v} - \frac{c_L(1 - c_L)u_G^2}{\left(v + \dfrac{c_L}{\rho_L \epsilon_L^*}\right)^2} \right)^{1/2},$$

where

$$-\frac{\partial p(c_L, v)}{\partial v} = \frac{r_G T}{1-\epsilon_L^*} \frac{1-c_L}{\left(v - \dfrac{c_L}{\rho_L(1-\epsilon_L^*)}\right)^2}.$$

Generally, we may assume that

$$-\frac{\partial p(c_L, v)}{\partial v} \gg \frac{c_L(1-c_L)u_G^2}{\left(v + \dfrac{c_L}{\rho_L \epsilon_L^*}\right)^2}.$$

Hence, the system [25]–[28] is hyperbolic with the characteristics

$$\lambda_1 = 0, \quad \lambda_{2,3} \approx -\frac{c_L u_G}{v + \dfrac{c_L}{\rho_L \epsilon_L^*}} \pm \left(-\frac{\partial p(c_L, v)}{\partial v}\right)^{1/2}. \qquad [29]$$

In the following we need also the right eigenvectors $\mathbf{r}_i$ of $\mathbf{A}(\mathbf{u})$, corresponding to λ_i, $i = 1, 2, 3$:

$$\mathbf{r}_1 = \left(1, -\frac{\partial P}{\partial c_L}\Big/\frac{\partial P}{\partial v}, 0\right)^{Tr}, \quad \mathbf{r}_{2,3} = (0, 1, -\lambda_{2,3})^{Tr}. \qquad [30]$$

5. THE RIEMANN PROBLEM FOR SIMPLIFIED EQUATIONS

Neglecting both frictional forces ($f = 0$) and mass transfer ($\dot{m}_L = 0$), we consider the Riemann problem for the system [25]–[28], i.e. the Cauchy problem with the initial conditions of special form:

$$\mathbf{u} = \begin{cases} \mathbf{u}^+, & q > 0, \\ \mathbf{u}^-, & q > 0, \end{cases}$$

where $\mathbf{u} = (c_L, v, u_G)$, and $\mathbf{u}^+, \mathbf{u}^-$ are constant states. For calculating the solution of the Riemann problem we need some preliminary definitions (Lax 1957) in terms of λ_k and $\mathbf{r}_k$, $k = 1, 2, 3$ (see [29]–[30]). The eigenvector $\mathbf{r}_k(\mathbf{u})$ is called *linearly degenerate*, if

$$\mathbf{r}_k(\mathbf{u})\nabla_{\mathbf{u}}\lambda_k(\mathbf{u}) = 0,$$

and *genuinely nonlinear in the sense of Lax*, if

$$\mathbf{r}_k(\mathbf{u})\nabla_{\mathbf{u}}\lambda_k(\mathbf{u}) \neq 0.$$

Recalling [29]–[30] we get in our case:

$$\mathbf{r}_1(\mathbf{u})\nabla_{\mathbf{u}}\lambda_1(\mathbf{u}) = 0. \qquad [31]$$

$$\mathbf{r}_{2,3}(\mathbf{u})\nabla_{\mathbf{u}}\lambda_{2,3}(\mathbf{u}) \approx \pm\left(-\frac{p_{vv}(c_L, v)}{2(-p_v(c_L, v))^{1/2}} + \frac{c_L}{v + \dfrac{c_L}{\rho_L \epsilon_L^*}}(-p_v(c_L, v))^{1/2}\right) + \frac{c_L(1-c_L)u_G}{\left(v + \dfrac{c_L}{\rho_L \epsilon_L^*}\right)^2}. \qquad [32]$$

It follows from [31] that the eigenvector $\mathbf{r}_1$ is linearly degenerate. The sign of the expression [32] is mainly determined by the first and second terms, which have opposite signs. But, using [28] we obtain after some algebra

$$-\frac{p_{vv}(c_L, v)}{2(-p_v(c_L, v))^{1/2}} + \frac{c_L}{v + \dfrac{c_L}{\rho_L \epsilon_L^*}}(-p_v(c_L, v))^{1/2}$$

$$= -\left(\frac{r_G T(1-c_L)}{1-\epsilon_L^*}\right)^{1/2} \frac{1}{v - \dfrac{c_L}{\rho_L(1-\epsilon_L^*)}} \left(\frac{1}{v - \dfrac{c_L}{\rho_L(1-\epsilon_L^*)}} - \frac{c_L}{v + \dfrac{c_L}{\rho_L \epsilon_L^*}}\right) < 0$$

for the values v, c_L such that

$$0 < c_L < 1, \quad v > \frac{c_L}{\rho_L(1-\epsilon_L^*)}. \qquad [33]$$

It must be stressed that inequalities [33] are equivalent to the inequality

$$1 > \epsilon_L > \epsilon_L^*.$$

Hence, the eigenvectors $\mathbf{r}_{2,3}(\mathbf{u})$ are genuinely nonlinear in the sense of Lax. The properties of linearly degeneracy and genuinely nonlinearity imply a very simple structure of the self-similar solution $\mathbf{u}(t, q) = \mathbf{u}(q/t)$ of the Riemann problem (Lax 1957). Indeed, for our case it is similar to the solution of the Riemann problem for gas dynamics equations. However, unlike the gas dynamics case, where one must use the Hugoniot adiabate, the Poisson adiabate works in present case, because the conservation of the "pseudo mass concentration" is the consequence of the mass conservation. To demonstrate an explicit solution, we consider the water hammer problem: to find the solution $\mathbf{u} = (c_L, v, u_G)$ in the region $t > 0$, $q < 0$ with the initial conditions

$$\mathbf{u}(0, q) = \mathbf{u}^-$$

and boundary conditions for $q = 0$:

$$u_G(t, 0) = 0.$$

This problem appears when one closes very rapidly the outlet of a pipeline which produces a shock. The Rankine–Hugoniot conditions for the system [25]–[28] are

$$\sigma[c_L] = 0, \quad \sigma[v] + [u_G] = 0, \qquad [34]$$

$$\sigma[u_G] - [P] = 0, \qquad [35]$$

where the sign [...] denotes the jump across the shock. If $\sigma \neq 0$, we get from [34]–[35]

$$c_L = c_L^-, \quad (u_G - u_G^-)^2 = (P - P^-)(v^- - v). \qquad [36]$$

We shall consider in [36] the specific volume v as a given function of the "pressure" P and the velocity u_G for the fixed value of c_L^- (equation [28]). For $u_G < u_G^-$ the equations [36] define, in the plane of variables (P, u_G), a continuous monotonic curve, passing through the state $\mathbf{u}^-$. The intersection of this curve with the axis $u_G = 0$ gives the "pressure" behind the shock wave. The velocity of the shock is defined from [34]–[35]

$$\sigma = -\frac{[u_G]}{[v]} = \frac{[P]}{[u_G]}.$$

6. SUMMARY

(1) A mass Lagrangian coordinate associated with the velocity of the gas is introduced for the drift–flux model. When we neglect both mass transfer and frictional forces, the resulting system is similar to the one-dimensional gas dynamics equations written in terms of the specific volume, the velocity and the entropy. Note that the pressure of this "pseudo-gas" depends not only on the specific volume and the entropy, but also on the velocity.

(2) We have shown that if there is no mass transfer, the equation for the "pseudo mass concentration" of the drift–flux model can be integrated independently in the mass Lagrangian coordinates, and this integration is valid even in the case of shocks.

(3) A simplified model typically used for slug flow at high Froude number in horizontal pipelines is considered and the solution of the water hammer problem to this model is obtained.

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